SPECIAL RELATIVITY

A. B. Lahanas
University of Athens, Physics Department, Nuclear and Particle Physics Section, Athens 157 71, Greece

Abstract
We give an introduction to Einstein's Special Theory of Relativity. After a short historical perspective, we discuss the Lorentz transformations and the measurement processes of lengths and time intervals. We also give an account of relativistic kinematics which are useful to accelerator physicists.

1. INTRODUCTION

This lecture is intentend to cover the basic notions needed for accelerator physicists giving emphasis on relativistic kinematics. The topics to be discussed are the following :

- From Galilean to Einstein Relativity
- The Lorentz Transformations
- Measuring Lengths and Time Rates
- Mathematical Formulation of Special Relativity
- Relativistic Kinematics
- Minkowski diagrams

The material presented here can be found in almost any standard textbook on Special Relativity. See for instance [1, 2, 3] and lectures given in previous CERN schools [4]. These references are only indicative and by no means can exhaust the rich literature on the subject .

2. FROM THE GALILEAN TO EINSTEIN RELATIVITY

Untill the end of the nineteenth century it was almost a doctrine that the laws of Classical Mechanics are invariant under Galilean transformations. Consider two inertial frames, S and S', the latter moving with velocity $\vec{V}$ relative to S. If their coordinates are $(\vec{x}\ ,\ t\ )$ and $(\vec{x}'\ ,\ t')$ respectively then these are related by

$$\vec{x}\,' = \vec{x} - \vec{V}\,t\ ,\ \ t\,' = t\ . \tag{1}$$

(We remind that in an inertial frame free bodies have no acceleration.) Note that time is assumed universal in the *Galilean transformations* given by eq. (1). This was postulated and was not based on any experimental information !
Then according to the Galilean Relativity

All inertial frames are equivalent, that is the laws of physics should be the same in any inertial frame.

As an example consider N point particles interacting with potentials that depend on their relative distances. Their equations of motion are

$$m_i\,\frac{d^2\,\vec{r}_i}{d\,t^2} = -\sum_{j\neq i}\nabla_i\,V(\,|\vec{r}_j - \vec{r}_i|\,)\ .$$

It is trivial to verify that these are indeed invariant under the transformations given by (1).
Although correct for the laws of Mechanics, this is not true for the laws of Electrodynamics (Maxwell) which are not invariant under (1) ! Thus if we accept their correctness, within the framework of Galilean

dynamics, we are also forced to assume then there is an absolute frame of reference, the **“Ether frame”**, in which Maxwell laws are valid and the velocity of light in *vacuo* is **“c”**.

Fizeau’s experiments, Aberration of light and the celebrated Michelson - Morley (1886) experiments were not reconciled with the **ether hypothesis**. An attempt to save the concept of the **stationary ether** was the Lorentz - Fitzgerald contraction hypothesis which states that *Bodies are contracted in the direction of their motion to the ether* according to the law

$$L \;=\; L_0 \sqrt{\left(1 \;-\; \frac{u^2}{c^2} \right)} \; .$$

This hypothesis contains the seeds of the **Special Theory of Relativity** .

2.1 Einstein’s postulates

Based on all experimental information available at the time, Einstein postulated that

- Laws of nature, including Electromagnetism, are the same in all inertial frames.
- The speed of the light is independent of the speed of its source.

If we accept this we need sacrifice the Galilean Relativity, or putting it in a different way, Galilean Relativity needs modification. As we shall see, Galilean laws of Mechanics are only valid for low velocities $(v << c)$.

3. THE LORENTZ TRANSFORMATIONS

Consider two inertial frames S , S' , moving with velocity $\vec{v}$ relative to each other. t , $\vec{x}$ are the coordinates in S and t' , $\vec{x}\,'$ are those in S'. For simplicity we shall assume that their origins coincide at $t = t' = 0$. Assume that a light signal is emitted from their common origin at this moment. Two observers, O , O' , who are at rest in the systems S , S' , both see a spherical shell to expand with the velocity of the light c. The equations of the wave fronts are

$$c^2\, t^2 \;-\; (\vec{x})^2 \;=\; c^2\, t'^2 \;-\; (\vec{x}\,')^2 \;=\; 0 \; . \tag{2}$$

The transformation laws connecting the coordinates of the two systems should guarantee the vanishing of the two expressions in eq. (2).

In order to simplify the discussion, suppose that the coordinate axes of the two frames are parallel and the frame S' moves with velocity $\vec{v} \;=\; (\, v,\; 0\;, 0\,)$ relative to S. The **Lorentz transformations** for passing from system S to system S' are the following,

$$\begin{aligned} x' \;&=\; \frac{x \;-\; v\,t}{\sqrt{1 \;-\; \frac{v^2}{c^2}}} \;,\; t' \;=\; \frac{t \;-\; \frac{v}{c^2}\,x}{\sqrt{1 \;-\; \frac{v^2}{c^2}}} \\ y' \;&=\; y \;,\; z' \;=\; z \; . \end{aligned} \tag{3}$$

One can verify that their inverses simply follow by interchanging $(\, t,\, \vec{x}\,)$ with $(\, t',\, \vec{x}\,'\,)$ and replacing v by $-\,v$ in eq. (3), as they should, since the two systems are equivalent. These transformations preserve the quadratic forms $c^2\; t^2 \;-\; (\vec{x})^2$ and hence the shape of the wave fronts. In matrix form, and supressing the y and z coordinates for convenience, the particular Lorentz transformation can be written as

$$\begin{pmatrix} c\,t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\,\gamma \\ -\beta\,\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\,t \\ x \end{pmatrix}$$

where $\beta \;\equiv\; v/c$ and $\gamma \;\equiv\; (\, 1 \;-\; \beta^2\,)^{-\frac{1}{2}}$. The consequences of these transformation laws, connecting the coordinates of the two systems, will be discussed in the following.

4. MEASURING LENGTHS AND TIME RATES

4.1 Comparison of clock rates

Suppose that we want to compare the clock rates in the two systems of reference discussed in the previous section. In each reference frame the events will be characterized by their time and position they occur. Consider the following events as seen by an observer O' who is at rest in the frame S'. The event $(t_e' \, , \, x_e')$ is emission of a light signal at the time $t_e{}'$ from the point $x_e{}'$. The event $(t_r' \, , \, x_r{}')$ is reception of the signal, after being reflected by a mirror, at the same point $x_r{}' = x_e{}'$ at a subsequent time t_r' . To an observer O who is at rest in S the same events have coordinates $(t_e \, , \, x_e)$ and $(t_r \, , \, x_r)$ with $x_e \neq x_r$. From eq. (3)

$$t' = \gamma \, (\, t - \frac{v}{c^2} \, x \,)$$

and thus the time elapsed between the two events, for the observer O', is

$$\Delta \, t' = t_r' - t_e' = \gamma \, (\Delta \, t - \frac{v}{c^2} \, \Delta \, x \,)$$

where $\Delta \, x = x_r - x_e = v \, \Delta \, t$. From this we get the following relation for the times elapsed as these are recorded in the two systems,

$$\Delta \, t = \frac{\Delta \, t'}{\sqrt{1 - v^2/c^2}} \quad . \tag{4}$$

This is the wellknown **Time Dilatation** formula. $\Delta \, t'$ is **"Proper Time"** for the observer O'. That is it is the time elapsed between two events occuring at the same place in system S', as read by a clock which is at rest in S'. What eq. (4) states is that *moving clocks run slower !* or phrased in other way

Viewed from any one system the clocks of any other system appear to be loosing time.

Atomic vibrations and course of life itself behave like clocks! If the observers O , O' are twin brothers and O' sets of a journey then O' must be younger when he returns ! We should point out that this argument discloses an inherent contradiction since O may be considered as the traveler and O' to be at rest. Then O is younger! This is the famous **Twin paradox**. However we should be cautious since during departure, turning and arrival gravitational fields develop which affect the beating of clocks ! To resolve this paradox we need know the basic notions of the **General Relativity**. (For an elegant discussion of the twin paradox see [1] .)

The time dilatation is verified in Particle Physics. The following example is wellknown. Charged pions have a half time $\tau \approx 2.6 \; 10^{-8} \; sec$. However they can penetrate the atmosphere travelling distances $h \approx 50 \; Km$, or even larger, and some of them reach the earth's surface. According to their lifetime they should travel distances at most $\tau \, c \approx 7.8 \; m$! Thus without time dilatation only a few $e^{-\frac{50,000}{7.8}} \ll 1$, that is practically none, should reach earth's surface. However τ is the *proper time*, that is the time as measured by a clock in pion's rest system. For observers on earth this time is larger, and according to the time dilatation formula is given by,

$$T = \frac{\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma\tau > \tau \; .$$

Therefore for observer's on earth, the minimum velocity required to reach the earth surface is $T \, v_{min} = h$ from which it follows that $\gamma \, \tau = \frac{h}{v_{min}}$. The latter yields $v_{min} = .999999987 \; c$, that is very close to the velocity of light and such enormous velocities do indeed occur in cosmic rays! From the point of view of the observer who is moving with the pion, the lenght h is contracted, as we shall see, by γ^{-1} and the distance to be travelled is $D = h \, / \, \gamma$. With v_{min} , the distance seen by the observer moving with the pion is $D \approx 7.8 \; m$, instead of $50,000 \; m$!

4.2 Measuring Lengths

Another consequence of the Lorentz transformation is the *length contraction*. Suppose that we want to measure the length of a rod which is moving with system S' having its ends at the points x'_A , x'_B. Its length, the **proper length** as is said, in S' is $L_0 = x'_B - x'_A$. In order to measure its length in S we need know where its ends x_A , x_B are, with respect the system S, at the same time $t_A = t_B$! From the Lorentz transformations (3) we have

$$x'_A = \gamma (x_A - v \, t_A) \, , \, x'_B = \gamma (x_B - v \, t_B)$$

Since $t_A = t_B$ it follows that $L_0 = x'_B - x'_A = \gamma (x_B - x_A) = \gamma \, L$ where L is the rod's length as this is measured in S. Therefore we arrive at the **Length Contraction** formula

$$L = \gamma^{-1} L_0 \quad . \tag{5}$$

According to (5) moving rod looks contracted by the factor $1/\gamma$. This is the **Lorentz-Fitzgerald** contraction mentioned in the beggining.

5. MATHEMATICAL FORMULATION OF SPECIAL RELATIVITY

In order to proceed further we need some mathematical definitions and tools which will prove to be extremely useful in the sequel.

For any event we define the *position four-vector* as the *quartet*

$$x^\mu \equiv (c \, t \, , \, \vec{x} \,) \, , \, \mu = 0,1,2,3 \quad .$$

According to it $x^0 = ct$, $x^1 = x,$, $x^2 = y,$, $x^3 = z$. Under a Lorentz transformation L this quartet transforms in the following way

$$x\,' = L \cdot x \; .$$

In this notation L is a 4×4 matrix and x , $x\,'$ are one-column four-row matrices. For instance if the systems S , S' have their x-axes parallel and S' is boosted with velocity $\vec{v} = (v, \ 0, \ 0 \ )$ the above transformation is

$$\begin{pmatrix} c\,t\,' \\ x\,' \\ y\,' \\ z\,' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\,\beta & 0 & 0 \\ -\gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \quad .$$

Any quartet $a^\mu \equiv (a^0, \ a^1, \ a^2, \ a^3)$ transforming the same way as the position four vector, that is like

$$a\,' = L \cdot a \quad ,$$

defines a *4-vector* which is referred to as a *contravariant 4-vector*. Given any contravariant 4-vector a^μ one can define its *covariant* a_μ , denoted by having all its indices down, by

$$a_\mu \equiv (\, a^0, - \, a^1, \ -a^2, \ -a^3) \; .$$

5.1 General Invariants

For any two vectors a^μ , b^μ their *scalar product* is defined by

$$a \cdot b \equiv a^0 \, b^0 - \vec{a} \cdot \vec{b} \; . \tag{6}$$

This is invariant under Lorentz transformations.

The *length* squared of a vector, Q^μ, is defined by

$$Q^2 \equiv Q \cdot Q = (Q^0)^2 - (\vec{Q})^2 \tag{7}$$

This can be either positive, and the vector is said to be **time-like**, or negative, and the vector is said to be **space-like**, or zero in which case we deal with a **light-like** vector.

The *distance* Δs between two events specified by the 4-vectors x^μ and $x^\mu + \Delta x^\mu$ is defined by

$$(\Delta s)^2 = \Delta x \cdot \Delta x = c^2 (\Delta t)^2 - (\Delta \vec{x})^2 . \tag{8}$$

This is certainly invariant under Lorentz transformations.
For any two events we can have the following three cases :

- If the two events are connected by a light signal then $(\Delta s)^2 = 0$ (light-like). This remains zero in any frame of reference reflecting the constancy of the velocity of the light.
- If $(\Delta s)^2 < 0$ (space-like) the two events cannot be causally connected. Light signal is not fast enough to connect these two space-time points !. The points are said to be not causally connected.
- If $(\Delta s)^2 > 0$ (time-like), then the two events can be causally connected.

As an example consider two events occuring at the same place in the system S'. Their distance is $\Delta s^2 = c^2 \, \Delta\tau^2$, with $\Delta\tau$ the proper time interval in S'. Since $(\Delta s)^2$ is invariant this equals to $c^2 \, (\Delta t)^2 - (\Delta x)^2 = c^2 \, (\Delta t)^2 - v^2 \, (\Delta t)^2$ or same $c^2 \, \gamma^{-2} \, (\Delta t)^2$, where now the coordinates refer to the system S . Therefore we have that $\Delta t = \gamma \Delta\tau$, that is the time dilatation formula (4).

It facilitates to define a 2-indexed quantity $g_{\mu\,\nu}$ with $\mu \, , \, \nu = 0, 1, 2, 3$, called the *Metric Tensor* , whose elements are

$$g_{00} = 1 \, , \, g_{11} = g_{22} = g_{33} = -1 \, , \, g_{ij} = 0 \ \text{ for } \ i \neq j \, .$$

(This definition of the metric tensor is in accord with what in the literature is often called *West Coast* metric). With the aid of the metric tensor we can express the previously defined quantities as it appears below,

$$a_\mu = g_{\mu\,\nu} \, a^\nu \ \ , \ \ a \cdot b = g_{\mu\,\nu} \, a^\nu \, b^\mu \ \ , \ \ (\Delta s)^2 = g_{\mu\,\nu} \, \Delta x^\nu \, \Delta x^\mu \ \ .$$

In these expressions summation is understood over the repeated indices.

In *General Relativity* the distribution of matter determines the form of the metric tensor, which follows by solving Einstein's equations, and $g_{\mu\,\nu}$ can be as given above only locally and far from massive bodies.

6. RELATIVISTIC KINEMATICS

In this section we shall deal with relativistic kinematics which is of primary concern to scientists working in accelerators.

6.1 Addition of velocities

We will first discuss how the law of the addition of the velocities is modified in the Special Theory of Relativity. Suppose that we have a reference frame S' which is moving with velocity $\vec{v} = (\, v, \, 0, \, 0 \,)$ relative to S. Moreover assume that a particle moves with velocity $\vec{u}$ in S and with $\vec{u}'$ in S'.
According to the Galilean Kinematics $\vec{u} = \vec{u}' + \vec{v}$. According to the Special Relativity we have

$$x = \gamma \, (\, x' + v \, t' \,) \, , \, t = \gamma \, (\, t' + \frac{v}{c^2} \, x' \,)$$

from which it follows that

$$u_x = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + v/c^2\ dx'} = \frac{u_x' + v}{1 + (v/c^2)\,u_x'} .$$

For the y-component $u_y = dy/dt = dy'/dt$, since $y' = y$ therefore

$$u_y = \gamma^{-1} \frac{dy'}{dt' + v/c^2\ dx'} = \gamma^{-1} \frac{u_y'}{1 + (v/c^2)\,u_x'} .$$

A similar result holds for u_z . These laws of addition can be also written as

$$u_{||} = \frac{u_{||}' + v}{1 + \vec{v}\cdot\vec{u}\,'/c^2} , \quad \vec{u}_\perp = \gamma^{-1} \frac{\vec{u}_\perp'}{1 + \vec{v}\cdot\vec{u}\,'/c^2} ,$$

where the symbols $||$, $\perp$ refer to the components parallel to and perpendicular to the velocity $\vec{v}$. From these we see that

- For low velocities, $u\,, v \ll c$, these reduce to the Galilean transformation laws.
- If in S' a light signal travels with velocity c the magnitude of its velocity in S is the same !
- $u'\,,\ v < c$ imply $u < c$. That is in any system the velocity of a real body can never exceed the velocity of the light.

6.2 Velocity, momentum and force acting on a particle

We will now generalize the definitions of the velocity, the momentum and the force acting on a point particle in the framework of the Special Theory of Relativity.

For a point particle whose position in space-time is given by the 4-vector $x^\mu = (c\ t,\ \vec{x}\ )$, its 4-velocity is the 4-vector defined by

$$u^\mu = \frac{dx^\mu}{d\tau} . \tag{9}$$

$d\tau$ is a proper time element, that is time as measured by a clock moving with the particle. The components of this vector are,

$$u^\mu = (\,c\,\gamma,\ \gamma\,\vec{u}\,)$$

where $\vec{u} = \frac{d\vec{x}}{dt}$ is the ordinary 3-velocity of the 3-dimensional space. Its length squared is a constant given by

$$u^2 = u^\mu\,u_\mu = c^2\,\gamma^2 - \gamma^2\,\vec{u}^2 = c^2 .$$

The 4-momentum is defined by,

$$p^\mu = m_0\,u^\mu \tag{10}$$

where m_0 is characteristic of the particle called *rest mass*. The components of the 4-momentum are given by

$$p^\mu = (\,c\,m\,,\ m\,\vec{u}\,) .$$

The mass m appearing in this equation is given by

$$m \equiv \gamma\,m_0 = \frac{m_0}{\sqrt{1 - u^2/c^2}} .$$

The length squared of the 4-momentum is

$$p^2 = p^\mu\,p_\mu = m_0^2 u^\mu\,u_\mu = m_0^2\,c^2 .$$

The 4-force, or Minkowski force, is defined by

$$F^{\mu} \; = \; \frac{dp^{\mu}}{d\tau} \; . \tag{11}$$

Its components are

$$F^{\mu} \; = \; (\; \gamma \; c \; \frac{dm}{dt} \; , \; \gamma \; \frac{d\vec{p}}{dt} \;)$$

where $\vec{p} \; = \; m \; \vec{u}$ is the ordinary three momentum .

Differentiating $p^2 = m_0^2 \; c^2$ with respect τ we get $p_{\mu} \; (\; dp^{\mu}/d\tau \;) \; = \; 0$, or same, $p_{\mu} \; F^{\mu} \; = 0$. This last equation can be also written as

$$\frac{d \; (\; m \; c^2 \;)}{dt} \; = \; \vec{u} \cdot \vec{F} \; . \tag{12}$$

In eq. (12) $\vec{F} \; = \; d\vec{p}/dt$ is the ordinary 3-force defined as the time rate of the 3-momentum $\vec{p} \; = \; m \; \vec{u}$. Equation (12), states that *the rate of doing work is the rate of changing* $m \; c^2$. Based on this Einstein asserted that the energy $E(u)$ of a particle with velocity u is

$$E(u) \; = \; m \; c^2 \; = \frac{m_0 \; c^2}{\sqrt{1 \; - \; u^2/c^2}} \; . \tag{13}$$

Then according to (13) the particle has a rest energy given by

$$E(0) \; = \; m_0 \; c^2 \; \; .$$

This is experimentally verified in inelastic processes in particle physics, such as the decay $\pi_0 \rightarrow \gamma\gamma$ for instance. In the frame where the pion is at rest the total energy of the produced photons equals to the pion's rest energy.

The kinetic energy of the particle is unambiguously determined from the difference $T(u) \; = \; E(u) \; - \; E(0)$, from which it follows that

$$T(u) \; = \; m_0 \; c^2 \; (\; \gamma \; - \; 1 \;) \; . \tag{14}$$

It is useful to remark that for low velocities, $u \ll c$, the energy and momentum of the particle can be approximated by their nonrelativistic expressions

$$\begin{aligned} E \; &\approx \; m_0 \; c^2 \; + \; \frac{m_0 \; u^2}{2} \\ \vec{p} \; &\approx \; m_0 \; \vec{u} \; . \end{aligned}$$

From eq. (14), relating the kinetic energy to γ, one can express the velocity as function of the kinetic energy T. The saturation of the relative velocity $\beta \; = \; u/c$ as function of the dimensionless ratio $T/m_0 \; c^2$ is plotted in the figure 1. This ratio is independent of the rest mass. One observes, from figure 1, that values of β that are close to unity are obtained for values of the rescaled kinetic energy $\hat{T} \; = \; T/m_0 \; c^2$ larger than about 2. For an electron (proton) this corresponds to kinetic energy larger than about 1 MeV (1.8 GeV).

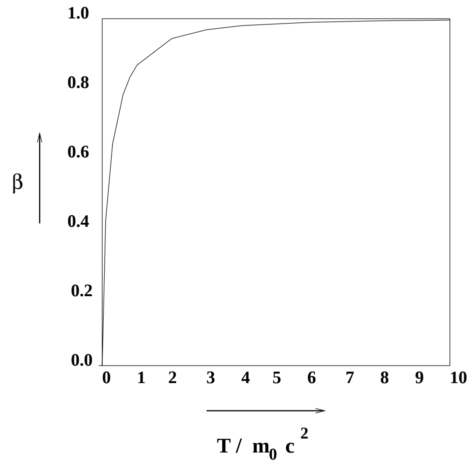

Fig. 1: The relative velocity as function of $T/m_0\,c^2$.

6.3 Energy and momentum conservation

Another way of expressing the four-momentum of a particle is the following

$$p^\mu \;=\; (\;\frac{E}{c}\;,\;\vec{p}\;)\;.$$

Using this, the relation $p^2 \;=\; m_0^2\;c^2$ takes the form

$$E^2 \;=\; c^2\;\vec{p}^{\,2}\;+\;m_0^2\;c^4$$

Then the conservation of the energy and momentum in particle collisions can be expressed by

$$\sum_i\;p_i^\mu\;=\;\sum_j\;p_j^{\prime\,\mu}\;. \tag{15}$$

In eq. (15) p_i^μ $(p_j^{\prime\,\mu})$ refer to the initial (final) particle momenta.

6.4 Relation between first derivatives

Particles in accelerators have their energies and momenta spread over a certain range $E\;\pm\;\Delta\;E,$, $p\;\pm\;\Delta\;p$. By differentiating we can derive relationships between corresponding variations $\Delta\;\beta$, $\Delta\;\gamma$ in the quantities β , γ .

In terms of γ the relative velocity is $\beta\;=\;\sqrt{(\;1\;-\;\gamma^{-2}}\;)$ from which it follows that

$$\frac{\Delta\;\beta}{\beta}\;=\;\frac{1}{\gamma^2\;-\;1}\;\frac{\Delta\;\gamma}{\gamma}\;.$$

From $T\;=\;m_0^2\;c^2\;(\gamma\;-\;1)$ it follows, by a straightforward calculation, that

$$\frac{\Delta\;T}{T}\;=\;\frac{\gamma}{\gamma\;-\;1}\;.\frac{\Delta\;\gamma}{\gamma}$$

From $E\;=\;m\;c^2\;=\;m_0\;c^2\;\gamma$ we also have,

$$\frac{\Delta\;E}{E}\;=\;\frac{\Delta\;\gamma}{\gamma}\;.$$

The magnitude of the 3-momentum is $p = m_0\gamma\, u = m_0\gamma\beta c$, from which we get $\Delta p/p = \Delta\,(\beta\,\gamma)/(\beta\,\gamma)$. Then using the equations above we get

$$\frac{\Delta\, p}{p} = \gamma^2\,\frac{\Delta\,\beta}{\beta} = \frac{1}{\beta^2}\,\frac{\Delta\,\gamma}{\gamma} = \frac{\gamma}{\gamma\,+\,1}\,\frac{\Delta\,T}{T}\,.$$

All relations between first derivatives are summarized in the table displayed in figure 2 (Bovet et al.)

	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta}$	•	$\frac{1}{\gamma^2}\,\frac{\Delta p}{p}$	$\frac{1}{\gamma\,(\gamma+1)}\,\frac{\Delta T}{T}$	$\frac{1}{\gamma^2-1}\,\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p}$	$\gamma^2\,\frac{\Delta\beta}{\beta}$	•	$\frac{\gamma}{\gamma+1}\,\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\,\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T}$	$\gamma(\gamma+1)\,\frac{\Delta\beta}{\beta}$	$(1+\gamma^{-1})\,\frac{\Delta p}{p}$	•	$\frac{\gamma}{\gamma-1}\,\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$	$(\gamma^2-1)\,\frac{\Delta\beta}{\beta}$	$\beta^2\,\frac{\Delta p}{p}$	$(1-\gamma^{-1})\,\frac{\Delta T}{T}$	•

Fig. 2: Relations between first derivatives.

Examples:

A) *Two particles with equal rest masses have total energy E in the Laboratory frame (Lab), in which one is at rest. In their center of mass frame (COM) their energy is E_{CM}. Relate the two energies.*
Solution
The quantity $s \equiv (p_1 + p_2)^2$ is invariant. In the COM frame we have $(p_1 + p_2)^\mu = (\,E_{CM}/c\,,\,\vec{0})$ from which it follows that $s = E_{CM}^2/c^2$. In the Lab frame $s \equiv (p_1 + p_2)^2 = 2\,m_0^2\,c^2 + 2\,p_1\!\cdot\!p_2$. When particle "1" is at rest $p_1^\mu = (m_0\;c\;,\;\vec{0})$ and $p_2^\mu = (E_2\;/\;c\;,\;\vec{p}_2)$. From these we get $p_1\cdot p_2 = m_0\,E_2$ resulting to $s = 2\,m_0^2\,c^2 + 2\,m_0\,E_2$. Equating the expressions for s in the two frames it yields $E_{CM}^2 = 2\,m_0\,c^2\,(\,m_0\,c^2 + E_2\,)$ or same

$$E_{CM}^2 = 2\,m_0\,c^2\,E$$

B) *A proton and an antiproton collide to produce a $W^+\,W^-$ pair. Find the minimum energy to produce the two W's in the Laboratory (Lab) and Center of Mass (COM) frames respectively.*
Solution
In the COM frame we have $s = (\,p_1 + p_2\,)^2 = E_{CM}^2/c^2$ where p_1, p_2 are the four-momenta of the proton and antiproton (p and $\bar{p}$). This equals to $(\,p_{W^+} + p_{W^-}\,)^2 = E_W^2/c^2$ where E_W is the total energy of the produced W's. Since $E_W \geq 2\,M_W\,c^2$ it follows that

$$E_{CM} \geq 2\,M_W\,c^2 \approx 160\;GeV$$

Thus in the COM frame p and $\bar{p}$ should each carry energy $E_{p,\bar{p}} \geq 80\;GeV$.
In the Lab frame, we have from the previous example that $E_{CM}^2 = 2\,m_p\,c^2\,E_{Lab}$. On account of the inequality given above we therefore get

$$E_{Lab} \geq \frac{2\,M_W^2\,c^2}{m_p}\,. \qquad (16)$$

The projectile proton (or antiproton) in the Lab frame has energy $E \;=\; E_{Lab} \;-\; m_p\, c^{\,2}$ from which on account of eq. (16) it yields

$$E \;\geq\; (\; 2\, \frac{M_W^2}{m_p} \;-\; m_p \;)\; c^{\,2} \;\approx\; 1.36\;\; 10^4\;\; GeV\; !$$

C) *A particle of mass* m_0 *in its rest frame has 4-momentum* $p' \;=\; (\; m_0\; c\;,\; \vec{0}\;)$. *Apply a Lorentz transformation to find its energy and momentum in a system in which the particle's velocity is* $\vec{v} \;=\; (\; v,\; 0,\; 0\;)$.

<u>Solution</u>

Since momenta transform as 4-vectors it follows that $p' \;=\; L(v)\; p$, or inverting $p \;=\; L(-v)\; p'$. $L(v)$ denotes the appropriate Lorentz transformation corresponding to the velocity $\vec{v} \;=\; (\; v,\; 0,\; 0\;)$. From $p \;=\; L(-v)\; p'$ we get in a straightforward manner

$$p_x \;=\; \gamma\; (\; v\; E'/c^2 \;+\; p'_x\;)\;,\; E/c \;=\; \gamma\; (E'/c \;+\; v\; p'_x/c\;)$$

However in the particle's rest frame $\vec{p}' \;=\; \vec{0}$ and $E' \;=\; m_0\; c^2$ from which it follows that

$$E \;=\; \gamma\, m_0\; c^{\,2} \;\;,\;\; \vec{p} \;=\; \gamma\, m_0\; \vec{v}$$

as expected.

6.5 MINKOWSKI DIAGRAMS

A convenient way to conceive the notions presented in the previous sections is by using graphs, with axes representing position and time. The use of such graphs was first introduced by H. Minkowski in 1908 and in literature these are customarilly referred to as Minkowski diagrams.

In any system S every event can be represented by a point $(c\,t, x\;)$ on the *space-time* plane, and a series of events constistutes a *world line*. The *light cone* lines $s^2 \;\equiv\; c^2\; t^2 \;-\; x^2 \;=\; 0$ divide the space-time into the *past*, the *future* and the *elsewhere* region as shown in in figure 3. Every point in the *elsewhere* region cannot be connected with a light ray to the origin $(\; s^2 \;<\; 0\;)$.

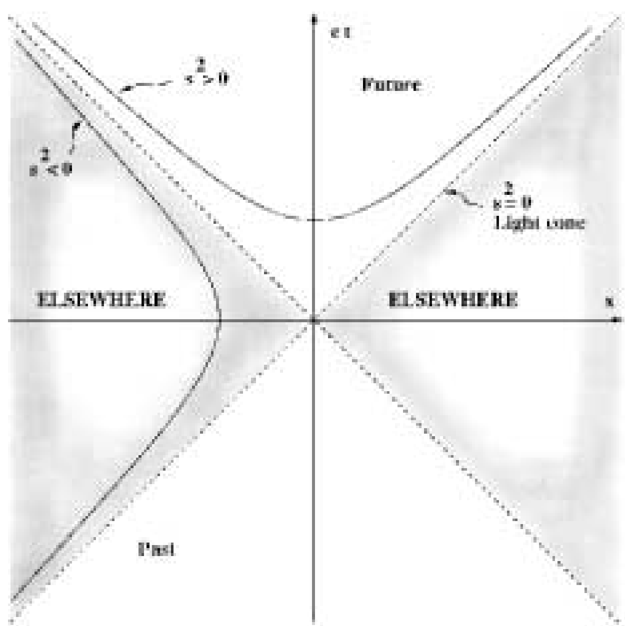

Fig. 3: A Minkowski diagram with the light cones displayed separating the space-time region into the *future*, *past* and the *elsewhere*.

Any other reference frame S', which moves with velocity v relative to S along the x-direction, having coordinates $(c\,t', x')$, is then represented by nonorthogonal axes as shown in figure 4. The slope of the axis x' is $\tan\,\omega = v/c$. We have assumed for simplicity that the y, z-axes of the two systems are parallel and that the origins of the two systems coincide at $t = t' = 0$.

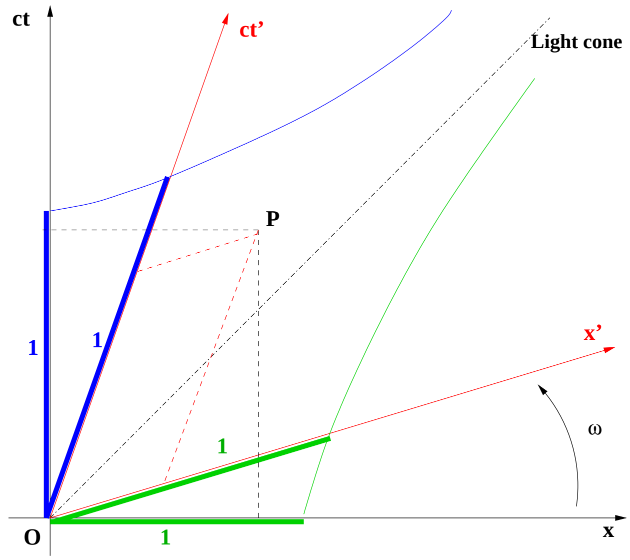

Fig. 4: A Minkowski diagram showing two different coordinate systems and the calibrating hyperbolas used to define unit distances.

Because of invariance under Lorentz transformations the hyperbola $c^2\,t^2 - x^2 = 0$ in S is also a hyperbola in S' system, i.e.

$$c^2\,t^2 - x^2 = c^2\,t'^2 - x'^2 = 0\,.$$

These *calibrating* hyperbolas can be used to compare lengths and time intervals. In figure 4 it is shown how the units of lenghts and time in x and $c\,t$ axes are projected onto their corresponding primed axes of the system S'.

Let us now see how the *time dilatation* and *length contraction* can be graphically represented by using Minkowski diagrams.

Consider first the time elapsed between two events as seen by two observers at rest in the two systems. The two events can be for instance emission of a light ray from the origin of the S' and arrival at the *same* point after being reflected by a mirror. The time elapsed with respect this system is $\Delta t'$ and is the length of the line segment (OC') divided by the velocity of light c, (see figure 5). For the observer at rest in S the coresponding time is Δt which equals to the magnitude of the segment (OC) again divided by c. Note that for this observer the emission and arrival of the light ray does not occur at the same place, hence Δt, unlike $\Delta t'$ for S', is not proper time for system S. To compare the two time intervals we draw the calibrating hyperbola passing from point C and see where this crosses the $c\,t'$ axis. The point of the intersectin is C''. Since the length of the line segment (OC'') is larger than (OC') we conclude that $\Delta t > \Delta t'$. Moreover from the geometry relating the coordinates of the two systems one can derive the time dilatation formula (4).

Let us now pass to discussing the *length contraction* using Minkowski diagrams. Suppose we want to measure the length of a rod which is at rest in the system S', having its ends located at the point 0 and L_0 as shown in figure 6. Its length in system S', the *proper length*, is L_0. In order to meausure

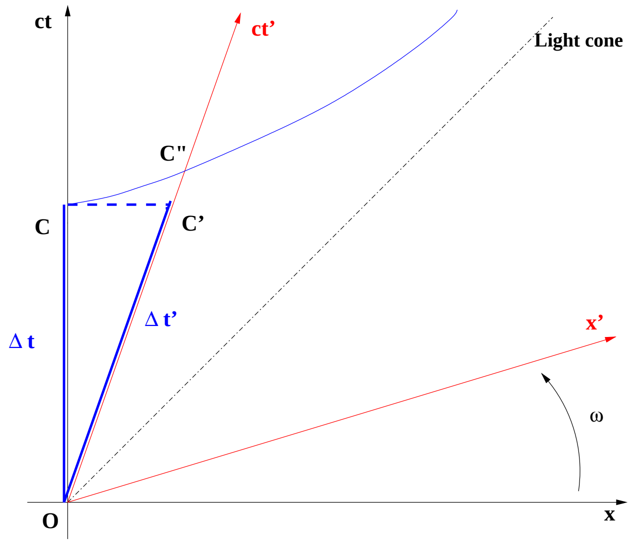

Fig. 5: Space-time diagram to illustrate the time dilatation (see main text).

its length in the system S we have to know where its ends lie at the same time with respect S. We see from figure 6 that with respect the system S the ends of the rod, at the time $t = 0$, have coordinates 0 and L. Therefore L is the length of the rod as measured by an observer who is at rest in S. Drawing the calibrating curve passing from the point L we see that it crosses the x' axis at L'. Since the length of the line segment (OL') is smaller than L_0 we conclude that the length of the rod, as is measured from an observer at S is smaller. Had we been more analytic, from the geometry relating the two systems we would have derived the *length contraction* formula given in (5).

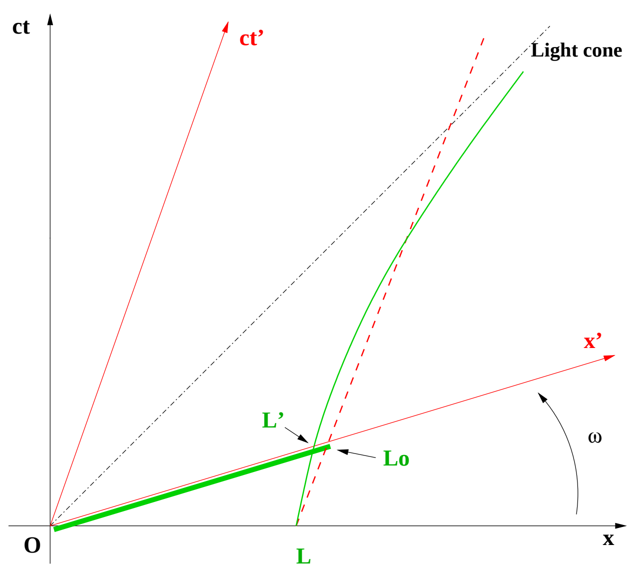

Fig. 6: Space-time diagram to illustrate the length contraction (see main text).

7. PROBLEMS

1. Two events occur sumiltaneously in a frame S and are separated by a distance of 1 Km along the x-axis. What is the time difference between these events as measured in a frame S' moving with constant velocity along the x-axis, if their spatial separation as measured in S' is 3 km?

2. Two trains, "1" and "2", each one of length 200 m in its own rest frame, travel in opposite directions. An instrument on train "1" can measure the times t_1' and t_2' the front and rear end of the train "2", pass by the position the instrument is placed at. If the time difference $t_2' - t_1'$ is $4.00 \times 10^{-6}\ sec$ find the relative velocity of the two trains.

3. A system $S\,'$ moves with velocity v_1, along the x-axis, with respect another system S. A third system $S\,''$ moves with velocity v_2 relative to the system $S\,'$. Find the Lorentz transformation connecting the systems $S\,''$ and S.

4. The phase of a plane wave is an invariant quantity under Lorentz transformations.
a) Use this to find the transformation law for the frequency and the wave vector.
b) A source emitting light signals of frequency ω_0 , moves away from an observer at rest. What is the frequency of the signal received by the observer ?

5. Prove that the phase space element

$$\frac{dp_x\ dp_y\ dp_z}{E}$$

where $E = \sqrt{c^2\ \vec{p}^2 + m_0^2\ c^4}$, is invariant under Lorentz transformations.

6. Find how the Electromagnetic Fields transform under Lorentz transformations, using the fact that the scalar and vector field are components of a contravariant 4-vector.

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