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Special relativity

1. Introduction

1.0. About this document

This aims to be the beginning of a book in progress on the foundations of mathematics and mathematical physics, with emphasis on geometry.

These are mainly classical subjects, but presented in a new approach, unrelated to any of the usual teaching programs.

It is especially intended for gifted students who do not only aim to succeed their exams but want to acquire deep understandings of those subjects.

1.1. What is mathematics ?

To say it briefly, mathematics is the study of possibly infinite systems made up of pure elementary objects, whose existence is abstract, independent of the exterior world. The only nature of each component (element, relation. . .) of these systems the fact of being exact, of the kind "all or nothing": two objects are either equal or different, are related or unrelated, a given operation gives exactly one result. . .

The choice of such systems to study is itself a mathematical choice, in the sense that it is given by conditions of limited complexity (unlike biology for example, which depends on arbitrary choices secretly accumulated by Nature during billions of years).

So, mathematical science is autonomous with respect to the circumstances of our world or universe, but can still apply to each of its so many places or particular aspects where such situations of systems of limited complexity can be met.

Though the nature of these objects is independent of any form of particular sensation or appearance, the only way to understand them is to represent them in some way. There can be several formulations or ways to represent a given theory, all being rigourously equivalent but their efficiency, their relevance can be very different. Most often, ideas are first developed in forms of more or less visual intuitions or imaginations, then cristallised into formulas and litteral sentences to be written, communicated or worked under these forms. And one can be freed from a given particular form of representation only by developing other forms of representation, and by exercising to translate objects and concepts from one form to the other.

1.2. What is geometry ?

Notion of geometry

Etymologically, the word *geometry* means "measure of the earth". Thus it is originally a science related to concrete reality: the one of the space around us, or the one of planes: the ground when it is flat, or any flat surface (paper, board. . .), that one can see and on which one can easily draw. The Greek geometer Euclid first wrote an axiomatic formulation of it in the case of the plane, thus giving his name to the usual (plane) geometry.

Euclidean geometry is the first *physical theory*, that is, a mathematical theory modelizing the physical reality, while idealizing it: whereas two points too close to each other can hardly be distinguished by physical measures, in mathematics one must split cases: they are either equal, or different. (then, between two different points there are other points, and so on: in a segment with finite length there is an infinity of points). And one chooses the simplest model: this theory is defined to be the simplest theory of the plane or space in agreement with usual physical experience. This is also the unique theory in which is exactly true any simple statement that seems true for this experience, up to measurement uncertainties.

For a modern mathematician, the name "geometry" is the family name carried by a large number of mathematical theories that have more or less parenthood together, like the fact it is more or less possible to visualize them, to draw figures of them. From a mathematical point of view, they are all equally "true", as studies of mathematical realities (systems of "points").

How can one visualize other geometries ?

But first, how can one visualize Euclidean geometry ? It is not the one we directly *see*, as the geometry of the visual field is the *spherical geometry*.

Indeed, the sight of a point only informs us on the half-line from the eye, to which this point belongs; and this half-line can be represented by its intersection point with a sphere centered on the eye. When one moves the eye and/or the head, the modification of what we see corresponds to a rotation of this sphere. The clearest example of this notion is the one of astronomical objects that we perceive only by their image on the celestial sphere (that rotates in appearance because of the rotation of the Earth).

But Euclidean geometry (of the plane or space) is only the one we are most used to *conceive*. Indeed, by seeing or imagining a two-dimensional image (on the vision sphere), we assign to each point a sensation of proximity or distance, and we are familiar with the way in which this perception is transformed when we turn the object or change the viewpoint with respect to it. It is this capacity, fruit of a mental construction, which constitutes our understanding of geometry; and our cleverness to handle it renders its use so easy like the one of an elementary sensation, so that its indirect nature becomes nearly anecdotical.

But, the intuitive perception of other geometries holds on the same principle: to see a thing for conceiving another. And to manage to conceive thus other geometries based on a vision which (unfortunately) remains the same, a good way is to work on this ability to conceive Euclidean space, by particularizing, modifying or generalizing it. (But then, it appears that this link to vision is not easily kept, and the conception work becomes more intense and shifting into formulas with a relatively weaker role of vision).

For example, one can conceive spaces (Euclidean or not) with dimension greater than 3. They do not mathematically bear any significant novelty in comparison to the line, plane or usual space, though the total absence of daily experience with them gives newbies the impression of an overwhelming difficulty.

To conceive these spaces, one can for example say that we just see two or three of these dimensions at once in the form of a cut or projection, but there are others present, we can change them...

Does Euclidean geometry exactly describe physical space ?

No. Not only is time added to the dimensions of physical space in special relativity theory to make up a 4-dimensional space (space-time) which is not Euclidean and ismore real (close to the physical reality) than our usual 3-dimensional space, but even if we restrict the consideration to its 3-dimensional appearance, the General Relativity theory teaches us that in reality, this space, as long as it can still be measured, does not exactly obey Euclidean geometry (independently of any problem of objects materializing or measurement accuracy). Of course, the error in assuming it Euclidean is extremely weak, much weaker than other errors much easier to understand and already often neglected: when we make a map of a region of the Earth (e.g. roughly shaped like a square), with diameter L, representing it by a region of a plane, the necessary error due to the roundness of the Earth is (as a ground distance) in the range of order of

$$\frac{L^3}{R^2}$$

where $R = 6370$ km is the radius of the Earth, which for example gives 2,4 cm for 10 km; the one due to general relativity, which thus remains in the tangent plane to the Earth, is of

$$\frac{2L^3 g}{Rc^2}$$

where $g = 9,8$ m.s^{-2} is the acceleration of gravity and $c = 3.10^8$ m.s^{-1} is the speed of light. It is thus 720 millions of times weaker (precisely, for any given region); in the case of a measurement of the whole Earth instead of a region, the error is of a few centimeters. So, given a choice of an "equator" as a perfect circle, its length is lower than its distance to the center times 2π (but the exact calculation of its distance to the center depends on the distribution of mass of the Earth among the differend depths).

If now one considers a vertical plane, another error appears, also stronger than the one from general relativity: the solid of length L that is turned to compare horizontal and vertical lengths is deformed by its own wheight (stretched if hanged, pressed if put), with an amplitude of about

$$\frac{L^2 g}{c'^2}$$

where c' is the speed of sound in the solid, which is necessarily lower than c according to special relativity.

On the other hand, the physics of the very small distances poses such problems that some theorician physicists seriously consider that the usual conceps of geometry do not hold anymore at the Planck scale (which is much smaller than the smallest distance for which the behaviour of particules is presently known).

1.3. What is mathematical physics ?

Current situation

There are two types of work in mathematical physics: fundamental physics and applied physics. The theoretical part of applied physics primarily consists in developing the consequences of fundamental physics using appropriate approximations (while sometimes using experiments when the approximations are too difficult to implement on a theoretical level).

The purpose of this work will be to present well-established and largely checked to date fundamental physical theories, from the simplest (mathematically simple and close to daily experience) to the most advanced (mathematically sophisticated and at the base of the explanation of the largest number of phenomena). But this order of arrangement is not strict. From the point of view of the "physical reality", only the last ones would have the status of fundamental physical theories as they are physically more fundamental and precisely in conformity with reality, whereas the first ones are only results and approximations of them; but the fact is that the teaching order of the fundamental theories (the order of comprehension in learning them and the intuitive meaning given to the concepts) is more or less the reverse of their dependence order as physical theories. (It is finally logical since if a theory A preceded a theory B both physically and pedagogically, one would classify B in applied physics).

First there is classical mechanics, the one of Galileo and Newton, which deals with forces and movements; the law of gravitation is formulated there simply. From it one can advance in two directions, which are the two types of modern physics which are opposed to classical physics: on the one hand Relativity (special then general), on the other hand, quantum physics and statistical physics, which are two parent theories closely related together. These theories will be introduced in more details in the corresponding chapters.

We will not specify the theoretical and experimental justifications which assess them, because these justifications are currently innumerable and among them, those at the historical origin of their discovery already lost any specific role.

Such a teaching approach deprived from matters of experimental justification is largely satisfactory since these well-established theories checked with an extreme precision, already include in their application field most of the realizable physical experiments. However, many of these experiments escape in practice this reduction because of the extreme complexity of their theoretical expression or of the too high diffulty of the resolution of the corresponding mathematical problems. This does not make fundamental physics a dead science, not only because the established theories are visibly incomplete (general relativity and quantum physics are very hardly compatible for reasons impossible to popularize), but quantum theory in its complete form compatible with special relativity (quantum field theory) appears, at the fundamental level, of a very high complexity lacking clear mathematical foundations, and the work of reformulation in the search of a better base remains active.

Philosophical aspects

Would we assume the physical objects have some "real nature", we could not in any case know what it is from experiment because all our perceptions pass by a translation from the external phenomena into the conscience we can have of them. Therefore, the best we can do is to study the physical world as a mathematical system, which lets us free to represent things in forms unrelated with those under which we usually perceive them. Let us explain that in details:

At least until now, it seems that any claim that an element of imagination would be more than another resembling the "true nature" of a certain element of physical reality is meaningless, because there is no means of comparing these natures : all our perceptions of the external world are indirect, converted by our body into nervous impulses, that the brain reprocesses to distinguish their global forms. Therefore, of the external physical world, only the structures, i.e. the relations between things, are accessible to our senses and can be the subject of our discussions, not any ontological nature. Anyway, how could a physical object have any identity of nature with a mental phenomenon ?

For example there is no resemblance of nature between the feeling of a color and the qualities of the objects seen, which are responsible for this color. This quality is due to the absorption coefficients of the various frequencies of electromagnetic waves by these objects, that determine the composition in frequencies of the light which the eye receives. Then, the cells of the eye only transmit a weak part of information from this composition (according to the properties of the molecules in the retina which do this work).

Also, we can hardly imagine time even just in accordance with the way in which we live it, which would show the nature of this lively experience of time beyond the mere form of the straight line which represents it mathematically: in no one moment can we imagine a duration as such, since one moment does not contain any duration; there are only future durations which do not exist yet, and past durations, which we perceive in our memory. But already, the memory of a duration is no duration any more, because it only represents this duration by a certain other instantaneous feeling.

Thus, any claim to describe things "as they are" would be meaningless, or then, it is no more physics but metaphysics, which can be possibly defended if announced clearly. Possibly, it can be useful when one works with a provisional theory being questioned to try to replace it with a more accurate one, that explicits subjacent structures to a given structure (for example, that is relevant to pass from the moles to the molecules, and from classical thermodynamics to statistical mechanics). But that only takes place in a relative and nonabsolute way, and is not the general case.

The object of Physics is thus to describe the physical laws, i.e. the expression as mathematical theories, of the obseved relations between the results of our observations of the physical world, since it is a general characteristic of mathematics to be focused on the structures, relations between the objects, and not on their nature. Then, between several mathematically equivalent presentations (translations) of a theory, that make various choices of forms of imagination to represent such or such aspect of reality, the best one is the one which makes it possible to grasp this mathematical theory in the easiest or most effective way.

The daily intuition of the physical world is adapted to the understanding of the problems of everyday life. But when one wants to come to study the structures of reality which appear in other contexts, it can be judicious to modify this choice. But in any case, a speech thus judiciously based on an unusual system of correspondences between real objects and forms of imagination used to represent them, is of course equally empty of significance regarding the true nature of these aspects or objects of physical reality than any other speech.

In lack of a complete theory of physics, we have partial theories, each of which deals with the properties of the observations in a certain experimental field of reality, i.e. a certain type of experiments with a certain degree of accuracy of measurements (depending on the parameters of the experiment). During the exploration of mathematical physics, we happen to consider a succession of theories corresponding to various experimental fields, aiming to be increasingly broad and with theoretical consequences more and more precisely in agreement with experience (or which gives more and more information to test).

Each one of these theories is first by nature a mathematical theory; its physical significance (its title of physical theory) then consists in allotting to each experiment of the considered experimental field a model in the theory, i.e. a certain subsystem of objects of the theory which represents mathematically the experimental device used.

But a theory has also an intuitive meaning, which lies in the choice of the form of representation best adapted to the comprehension of the theory. In this exploration, many geometrical theories intervene, not only to account for the behavior of the matter evolving inside the space which is familiar for us, but also to replace this space itself by other ones (this last case does not often happen: mainly in Special Relativity, General Relativity and what includes quantum gravity). And, among our usual intuitions or abilities to perceive the physical world, vision is the one which is clearest mathematically, and easiest to work on to be able, with it, to conceive many of these geometrical theories.

The difficulty of the newbie to question the form of his perceptions also arises in terms of the principle of reality, whereby "physical reality exists" whether we perceive it or not, in the name of what he tries to continue to think things in terms of "real" or "concrete" objects such as he usually perceives them.

However the problem is not to know whether reality contains or not other objects than those of our experiment or perception, because in any case the physical theories with which we work present some. The problem is to knowing which ones. The newbie imagines these objects of the underlying reality, which he cannot perceive directly, as resembling those which are familiar to him. In principle, he is not always wrong this way. Indeed, insofar as such or such phenomenon that he needs to explain can be due to mechanisms located in an experimental field broader than his daily experience, but not yet so broader that his intuition ceases being valid, his unability to directly observe these mechanisms in terms of this same intuition may be a mere matter of circumstance remediable by convenient apparatus (microscopes, telescopes, etc).

On the other hand, when while going further in experimental fields we are confronted with a more fundamental impossibility to refine the measuring instruments to observe the same type of explanatory real mechanisms, it is time to consider that there may be a good reason for this: namely, that there is no underlying reality resembling what the newbie would like to believe, because reality is of another form which can be understood only by different means, in different terms.

1.5. A few words of Euclidean geometry

Let us evoke here the Euclidean geometries of dimensions 1, 2 and 3. The one of dimension 1 (of the straight line) is mathematically simplest; the two following ones (of the plane and space) are almost of equivalent complexity, and will be treated here in the same manner.

Our vision is accustomed to understand the plane and space, but is a little broad to speak about the straight line: which line? horizontal, vertical...? One is tempted to place this line somewhere. One is already less tempted to situate the plane in space, since the two dimensions of the plane fill the two dimensions of our field of vision, not letting us see the vacuum around it. But whatever can be our modes of representations and possible relationships with our physical Universe, each one of these geometries, as a mathematical theory, is limited to consider constructions than can be made inside its own system (the line, the plane, the space) independently of any consideration of a possible external space in which this system could be situated. Even if such a consideration can be useful for intermediate calculations and reasoning or as an intuitive support, it is not regarded as belonging to the core realities of this theory, and must thus disappear in the objects of study and the final results. This independence of each geometry clearly appears in the example already evoked of the field of vision, which has the shape (or geometry) of a sphere: this sphere certainly has a well-defined center (the eye) but does not have a radius, it would be wrong to define it as a particular sphere situated in space. (It is the set of all half-lines from the eye.)

For this reason, it is wise to provide to the disposal of our intuition some forms of representations other than vision to figure out the geometry of the straight line (or geometry of dimension 1). Here are two. First, the feeling of time. Then, the concept of wire (infinitely thin), as a mechanical object whose nature is preserved whatever its configuration in space (straight lines, curves, broken lines...). Now let us present an intuitive introduction to Euclidean geometry.

Consider a wire of finite length, stopped in two ends (it is cut there, or its extensions are forgotten as if they did not exist). We define a *curve* as the image of a wire, the subset of the plane or space it can take. Each wire is characterized by its length, which is a positive quantity. For each curve, the wires that can be laid out on it are all those which have a same length, that one then calls the length of the curve. Any point of a wire divides it into two parts, whose sum of lengths is equal to the length of the first wire. The length of a curve is also measured by the time taken to follow this curve at a fixed speed (so are related these three unidimensional intuitions: wires, time and curves).

Marking the wire with a regular graduation to measure the lengths from an end (or more generally from any given point of the wire), or in an equivalent way, counting the times during its following at constant speed, provides a marking of each point of the curve by a number or quantity called its *affine parameter*.

1.6. Experimental foundations of affine plane geometry

We will define the affine geometry of the plane by the following experiment.

Let us observe from far away a figure drawn on a plane; we do not know (and will never know) at which distance this plane is, nor how it is laid out with respect to us (straight or skew; we do not see its border), nor if we see it from front or from behind (or equivalently, through a mirror). Various such observers will be always even unaware of how they are positioned in space the ones with respect to to the others.

We only know what we see, and its apparent size is small (it could hold in the lunar disc for example): we can take it in photograph, and make geometrical measurements on the photograph. The mapping from the real figure to the photograph or figure seen is called an *affine transformation*.

In these conditions, which are the concepts of geometry (properties or measurements of a figure – this will further be called the *structures*) whose appearance from any point of view of this type is always faithful to reality, up to measurement uncertainties (neglecting the effects of distortion related to the size of the object with respect to its distance) ? We call *affine plane geometry* the geometry of the plane* reduced to the study of the concepts which appear accurately in this experiment. It thus differs from the Euclidean geometry to which we are used, by the fact that its language is reduced: when two observers placed from different points of view communicate their observations of the same figure by phone, while having only the right to express themselves in this language, they will always agree. And they can even introduce new terms, as long as they can define them entirely by the only means of this language, they will still always agree.

Thus, all the concepts of affine geometry are concepts which also exist in Euclidean geometry. But as certain usual concepts of Euclidean geometry are no more valid, we need then, to be still able to work, to use instead some less usual concepts which remain in affine geometry, that can correspond to these Euclidean concepts for particular observers.

The same object of affine geometry admits multiple possible interpretations (or measurements) in the language of Euclidean geometry (each observer has his own interpretation defined by what he sees), and no measurement or property which only uses the affine language can express how to distinguish "which is the true one". Anyone can be regarded as true from the corresponding point of view.

The concept of straight line belongs to affine geometry. All concepts that can be defined from concepts of affine geometry, also belong to affine geometry. For example, the following concepts belong to affine geometry: alignment of points, half-lines, segments, parallels and parallelograms, lengths ratios of parallel segments, ratios of surfaces, barycentres, translations and homotheties, parabolas, as well as ellipses and hyperbolas with their centers.

In fact, the concept of straight line is sufficient to define all the other concepts of affine geometry, i.e. which do not depend on the observer in the experiment above. Therefore, any transformation which maps straight lines into straight lines is an affine transformation.

The following concepts do not belong to affine geometry, as observers in different places can disagree on them: distances, lengths of curves or segments, angles, orthogonality, circles, rotations, scalar product, axes and focuses of conics.

For example, if one sees a circle, someone else will not see a circle but only an ellipse. This can serve as a definition of the concept of ellipse: an ellipse is a figure obtained by an affine transformation

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* Here is why one can speak about the (infinite) plane although the figure is supposed to hold in the lunar disc: it is enough to start from much smaller figures than the lunar disc (one thus needs a telescope to observe it), from what the lunar disc will seem almost infinitely large like a plane: we can freely continue there the figure. . .

from a circle. In the same way, isometries as defined from Euclidean geometry are affine transformations which in affine geometry are no more distinguishable in larger sets of affine transformations of the plane.

We saw in Euclidean geometry the configurations of wires in the plane, which are certain maps of a wire into the plane. Here, it is no more necessary to distinguish them inside some larger set of maps from a wire in the plane, for example the one of *parameterized curves* (the wire is replaced by an elastic; if we keep the intuition of time to imagine the wire, the speed of travel along the curve can then vary in any way, different from one point of view to another).

2. The strangeness of Relativity theory

This introductory chapter to relativity and its strangeness aims to analyze the methodology on which usual introductory courses to special relativity traditionally rests, and to justify the adoption of a new approach better reflecting the use of this theory in a broader context of theoretical physics. It is particularly addressed to those having seen the usual presentation, but should be at least partially understandable by all.

It is possible to directly come to the core of the subject by skipping this and going to chapter 3.

2.1. Its name and some other exterior aspects

Relativity is in two stages called “special” and “general”. Special relativity theory is a theory of space and time that replaces the one previously stated by Galileo and Newton, by reducing it to the status of an approximation valid when relative speeds of the studied objects are negligible compared to the speed of light. (We will imply sometimes the adjective “special”, as we will not study general relativity in this text).

A consequence of relativity is that no signal, no information can be transmitted faster than the speed of light in the vacuum (which is also the one of radio waves), no more than it can arrive before it left, for reasons of non-contradiction of reality.

This speed is a universal constant $c \approx 3.10^8\,\mathrm{m.s}^{-1}$, which means precisely that what sends a signal at this speed, in the form of radio waves for example, to a point located at a distance of $3.10^8\,\mathrm{m}$ where it is immediately sent back, will have to wait two seconds after the first emission before receiving the return signal.

One should not be mistaken by the name of “relativity” which badly reflects the meaning of this theory: some authors of the theory regretted it thereafter, thinking that the name of “chronogeometry” for example would have fit better, but it was too late. The caricatural slogan “all is relative” had already taken possession of popular imagination, but its meaning was neither new nor correctly representative of the theory. Indeed, on the one hand the Principle of relativity was already contained in the Galilean theory, the only difference being that the change of reference frame affects a larger number of parameters (for example time) which describe given physical objects, in relativity theory compared to galilean theory (and this principle is thus recovered after electromagnetism seemed to contradict it). In addition, this slogan is likely to cause for example psychological inhibitions against the acceptance of the objective and measurable character of the rotational movement of an object (gyroscope, Foucault pendulum. . .).

Special relativity was discovered by different researchers: Poincaré, Lorentz, Minkowski, Einstein. General relativity was discovered by Einstein and Hilbert; it includes special relativity and gravitation (which it interprets as not being a force but the effect of the curvature of space-time. . .).

Between the two we can place electromagnetism, for its level of complexity (general relativity does not rigourously depend on it but uses about the same mathematical tools). General relativity cohabits naturally with electromagnetism : the electromagnetic field and the gravitational field interact in a coherent way while remaining distinct objects.

One can notice a mathematical tool which crosses a large number of physical theories: tensor calculus. To summarize, this formalism consists of systems of general operations of “multiplications” between vectors that can belong to different vector spaces. It is expressed in all its extent in quantum field theory where any tensorial expression that one can write from certain objects (given by the types of available particles) is realized by the corresponding Feynman diagram.

Its absence from the usual courses of electromagnetism is in fact an awkwardness: certain calculations (in classical mechanics and electromagnetism) use vectorial formulas or other rules of transformations of expressions which seem then of very mysterious origin or require complicated demonstrations, but would become simple and natural once translated into tensorial expressions. What makes this situation last is that it does not seem to currently exist in the literature a sufficiently clear introductory presentation of tensor calculus so that one can seriously consider introducing it on this level of teaching.

Historically, electromagnetism was developed before special relativity, and currently electromagnetism is still taught first in the first years of university. By observing it implies that the speed of light is constant, it later serves as a justification to develop special relativity. But this approach inherited from history, perpetuated by inertia, lack strongly of elegance (without relativity, the electromagnetic field is presented in the form of two objects, electric field and magnetic field, obeying the four Maxwell's equations which seem quite unnatural, whereas one or two tensorial equations on only one object is enough in the context of relativity). It would be more elegant and effective to start with special relativity and tensor calculus, if only suitable approaches of these theories were findable in the literature.

Electromagnetism within the relativistic framework is also called *classical electrodynamics* (in opposition to *quantum electrodynamics* which treats this force in quantum physics). One can conceive this theory as an extension of electrostatics, in the following way.

There is a formal similarity between the theories of electrostatics and magnetic induction (in three-dimensional space) which correspond by replacing the clarges of dimension zero (densities) by charges of dimension 1 (currents). Once noticed in terms of tensor calculus, it is enough to reexpress this same theory, formally independent of the choice of a particular dimension, to the case of relativistic space-time, to obtain classical electrodynamics. The magnetic field and the electric field are only the two components of the same field split relatively to the chosen inertial observer.

Classical electrodynamics presents the strange defect to be unable to regard the electrons as infinitely small material points in all exactitude without absurd consequences, but only according to approximations, giving up the idea to define their positions with a precision finer than a distance of the order of the *classical radius** of the electron $r_e = 2,82.10^{-15}$ m. However, this size is quite lower than the famous "uncertainty" of quantum physics (within which this problem is solved but in a very complicated way).

2.2. The origin of its paradoxes : the Galilean intuition

Seen from the top of a tower, a car can move away any far, indefinitely; it seems to reduce and to slow down as it approaches the horizon, although actually nothing special happens to it at its approach. It could never reach this horizon (except because of the roundness of the earth), nor even less go further. These facts do not astonish anybody. Why can then one be astonished by the following facts which however are but similar to them: a traveller can accelerate as long as he wants without anything happening to him, while someone else who would claim himself motionless would thus see him only go to speeds arbitrarily close to the speed of light, without ever reaching it, but while having with this approach some distortions, among which a deceleration of time and a contraction of lengths in the direction of his speed (which in fact does not even visually appear as a contraction but as a mere rotation; see details later) ?

This astonishment, which expresses a legitimate need to understand, is due to our natural tendency to think the problem in terms of what (unless you have a better suggestion) we will call "the galilean intuition", that we will now explain.

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* Comparable with the size of the atomic nucleus, it is the double of the radius of the sphere outside which the energy of the electric field calculated classically reaches the energy of mass mc^2 of the electron: seen at a lower scale than that, if the electron were a point, by withdrawing from it this energy and thus this mass which is in the surrounding field, the remaining mass would be negative, defying the laws of the mechanics… but in fact, the mass energy of the magnetic field due to the spin of the electron is still more significant near the scale of this classical radius and would thus give a larger radius.

We perceive the world in the form of images received by the eyes and interpreted instantaneously by the brain in three-dimensional images. There is thus a continuous succession of images, or in other words an image in evolution. In the same way in the theoretical activity, one imagines figures or images which follow one another or evolve during time.

The galilean intuition is then the fact of using this temporal dimension of our imagination, this faculty to let evolve in our imagination the images during time, to represent the temporal dimension of the physical universe we try to understand.

This mode of representation was adapted to the Galilean theory, which authorized the instantaneous distant interactions and in which the speed of light was infinite. Indeed, the image which an observer receives from the world at one time corresponded then to the image that another observer far away from the first (and that could be moving relatively to him) also receives at a certain time (thus at the "same" time), by a simple geometrical transformation (an displacement). It is thus a 3-dimensional image which the theorist can imagine at a certain time, and to which he can give an objective reality.

But that becomes false in relativity, owing to the fact that the speed of light in the vacuum is finite and the maximum possible for information. Therefore, galilean intuition is no more adapted to conceive this theory.

2.3. The methodological trap of its current teaching

The majority of special relativity courses, even recent ones, that can be found on the market, still more or less use the same approach faithful to the history of its discovery, far from any attempt of renewing considered impossible or without object, as this method was repeated more or less just the same since nearly a century: in the name of the sense of experimental principles and realities, they start by regarding as a core reality the type of appearances of the world perceived by the miserable points in the universe which we are. Being thus locked up in the galilean intuition, they then can only undertake to grasp therein also at any costs the reality of the physical space-time.

As the correspondence between physical reality and Galilean intuition that observation naturally gives is not satisfactory, they thus define an artificial one more satisfactory for the mind, called "inertial frame", by means of a virtual, complex and practically unrealizable experimental device. This concept of inertial frame which was inherited from galilean theory, will finally be a mathematical tool equivalent to the concept of an orthogonal frame in Euclidean geometry (or sometimes to the data of only one base vector of these frames, namely the time vector): useful notion to the study of geometry (but not absolutely necessary), and diverted here from its basic meaning into the call for an unsuited intuition. (Indeed, it declares instantaneous independent events, giving to "to see" in imagination a "present" distant event whose light will arrive only later, as if one could see a light signal coming before it reaches us. . .) (In galilean theory, one could also consider non-inertial reference frames without too many complications, which is no more the case in relativity: the accelerated material systems that can also be considered by the theory, and which are meant to define these other reference frames and did it well in galilean theory, are a worse point of view for the Galilean intuition in relativity; such a use of this intuition would mathematically correspond to the use of curvilinear frames of reference.)

Naturally unable to articulate this intuition in accordance with the theory to be conceived, they declare that the theory can be understood only by the rigour of the formulas and proofs, while rejecting any intuitive inspiration. But at the same time, in the name of the meaning of realities, they hopelessly go on clinging to this intuition. Because formulas can't exist in a pure state without losing any significance, they can only be relations between parameters which correspond to reality, which is defined as the object of our senses and thus of our intuition, yes but which ones and according to which correspondance ? It is this unfortunate choice of an arbitrarily fixed mode of representation, neither faithful to the experiment in its relation to reality, nor adapted to the internal understanding of the theory to conceive, which determines the appearance of the "fundamental formulas" and their complexity.

Then, fault of being able to give an intuitive physical meaning to a formula, they let it the authority that comes from the lengthy proofs for the fact that there cannot exist any other theory of space-time or mechanics (compatible with some principles expressed in experimental terms), however twisted it may be, in which it would not be rigorously true. Thus, in the name of the meaning of physical realities, they prove the unicity of the theory without giving the means to feel or really understand it, while almost forgetting to justify its existence (i.e. its non-contradiction), satisfied for that to note that the universe exists. (From where comes the proliferation of pseudo-revolutionary physicists who claim to refute relativity by simple thought experiments, presenting it as an absurd and poor interpretation of the famous experiences).

2.4. For a new approach of the theory

We will approach the relativity theory by an intuitive presentation which consists in using, instead of the Galilean intuition, a purely geometrical intuition.

Here is the reason. We perceive dimensions of physical reality in the form of three sensations : visual images (of dimension 2), depth and time; they naturally provide three forms of imagination for the theorist. We recognized that the three dimensions of space have physically the same nature and are in continuity between them in spite of their difference of sensitive form (vision and depth). This identity of nature and this continuity appear intuitive when by a rotation of the object the depth is exchanged with one of both visual dimensions. Then, let us generalize this reasoning by supposing that the time dimension itself also has physically the same nature as the three others, and let us make this fact intuitive by also representing time by a visual dimension of imagination (in fact this identity of nature is not complete but this approach is no less relevant, and the difference will be specified later).

One could be tempted to reproach this method the absence of rigorous justifications founded on reality (statements of physical principles resulting from experimental observations) to build and prove the theory (or to state it more clearly, the fact that it is not based on the Galilean intuition). But its aim is to give a clear vision of things without embarrassing oneself at the beginning by their experimental appearances. The coherence and the consequences of this vision will naturally show that it describes a possible world compatible with the famous physical principles. The readers interested by the proof that there is no other possible world, apart from special relativity and the galilean theory, are thus invited to refer for this to the traditional courses on this subject. (We can also notice that indeed, general relativity precisely describes another world closer to the truth. . .)

One could also reproach it to not being simpler, and even be more difficult to understand than the traditional method. Already, people accustomed to the latter (or those who having started to learn it would like to have it explained better to them), are likely to be reluctant to restart everything from scratch by changing the approach completely. But perhaps even some, according to their way of thinking, could really better manage with it. Indeed, it had somehow the advantage of making it possible to formulate the theory and solve problems without really doing the work to understand it and come into it: “it suffices” to write the proofs and to apply the formulas. However, this advantage which one can find until the formulation of the theory is likely to turn into a handicap later, when in front of a concrete problem one is lost in front of these formulas which one does not always know how to apply or handle, or whose use would requires pages of calculation.

Lastly, one could be tempted to think that it is not complete, omitting the devoted formulas considered necessary to the exact resolution of the practical problems (by often confusing the practicality of a problem with the arbitrary use of inertial frames and the associated vocabulary and imagination in its formulation). But finally, it is not necessary to develop so much specific formulas as if it were a completely new theory to our basic knowledge, because the essential part of relativity is a geometry indeed very similar to the Euclidean geometry (as it will be explained later): it is thus enough, in order to face the majority of problems, to reuse the various tools of study of Euclidean geometry, sustained by the corresponding intuition, and to specify how they are modified in the new geometry.

But in fact, the viewpoint that the following presentation aims to express, is not really new, as appears in the many further developments of theoretical physics that followed the historical discovery of relativity. Indeed, we can observe that the heavy formulas (“Lorentz transformations” and what follows) allegedly constitutive of relativity, while remaining true, do not explicitly appear as such any more in the chapters of general relativity, classical electrodynamics or other theories based on special relativity. Instead, one uses a simple object (a “metric” called “pseudo-Euclidean”) that testifies the real assimilation of this geometrical nature of space-time in the mind of physicists. It is for them so natural that it is useless to re-explain this fact, while the most interesting thing is to go further in its use. Therefore, the approach which will be proposed here would be new not in itself, but only *relatively to the usual universe of university education* .

Then, why did such a renewing of its teaching never occur since so many physicists assimilated it so well ? The answer is simple, but would just look strange to our world accustomed to judge the knowledge of pupils and students according to their capacity to write proofs and exercises according to the established rules: it is that a huge gap may sometimes separate the clear personal assimilation of a knowledge, from the capacity to find the right words to translate it into a clear and transmissible verbal form.

3. New presentation of special relativity theory

3.1. The core logic of the theory

According to the preceding explanations, the best way of understanding space-time is to start by giving up the attempt to understand it as such, in order to be able to quietly assimilate the concepts which constitute it as if it was unrelated, to not depend on background motives that would be counterproductive. The fact that special relativity has the title of a physical theory is equivalent to saying that the imaginary world which we will now present mathematically corresponds to the space time of our universe in the considered approximation field. But this fact should not be considered at first, as it does not bring anything to the understanding of the theory itself. It will suffice at the end, when all will be clearly understood, to simply notice: "And why not?".

So let us come to the conceptual contents of relativity. Here it is:

Imagine a world where all material things are fixed, in equilibrium.

Can you imagine imagine that? You will say: of course, it is too easy, this kind of situation is a particular case of the daily experience of all of us, which is more complicated. But special relativity must necessarily be something still more complicated, else it would have been understood since a long time ago, wouldn't it?

Admittedly, it is not exactly sufficient to imagine a motionless world, but mathematically the difference is relatively unsignificant. The essence of the complexity of things is contained in this particular type of daily experience.

This experience is mathematically not a trivial thing at all. Indeed, to describe the configuration of the various objects in equilibrium, it is necessary to make use of the language of geometry. However, the Euclidean geometry is in fact a mathematically complex theory, which one could not easily explain in simple and purely logical manner without starting from the basis of visual experience (for example to a blind man which would first not have any experience of space, if there were one). In the same way for the concept of equilibrium. But we are lucky, these two things are familiar to us. It only remains to add to it the three following points, which can each be introduced independently of the others, and to specify certain particular interactions between them.

– The relation of this story to space-time perception given by experience

– The change of geometry: the space in which things are obeys a geometry different from the one which is familiar for us; thus, it needs to be defined.

– The mathematical expression of the mechanics of equilibrium and the mechanical properties of particles.

This difference in geometry is weak in itself, as the new geometry is not particularly more complicated than the Euclidean geometry, but we are so much conditioned by our practice that this difference can seem to us extraordinary and insurmountable.

More precisely, there are two differences in geometry. The first is the dimension: it is 4, whereas we are accustomed only to dimensions 2 and 3. The second one is that it is not here the 4-dimensional Euclidean geometry (which automatically generalizes those of the usual plane and space). But it is another geometry which has strong similarities with the Euclidean geometry, and some differences.

According to a certain use, we will call it the pseudo-Euclidean geometry. Like the Euclidean geometry, it exists in any dimension $n \geq 2$.

There are approximately three possible ways to introduce it. The first would be to start from physical principles among which the invariance of the speed of light. It hides its major analogy with the Euclidean geometry, by privileging the structures which are specific for it, related to the cone of light (which we will see at the end), against the structures which are in common. A second method consists in giving a mathematical construction of it, starting from affine geometry, in a way similar to the construction of the Euclidean geometry (the most convenient way to do this is probably the introduction of a scalar product on the associated vector space).

As these methods are easily found elsewhere, we will present here a third method (not that it is better in the absolute but it also has specific advantages), which consists of a kind of "magic" (nonrigorous) transformation which makes the correspondence between Euclidean geometry and pseudo-Euclidean geometry. Its advantage is that it does not require to go back to the mathematical fouundations of geometry, but it makes it possible to re-use the naive knowledge and general methods available in Euclidean geometry, by applying just a quite simple translation of the results to obtain the corresponding ones in pseudo-Euclidean geometry. It thus has the advantage of enabling us to transport directly in the new geometry the very diverse intuitions which we could inherit from experience in connection with Euclidean geometry without requiring their mathematical reconstruction, advantage that we will use later.

This method is mathematically expressed by the concept of *analytical extension*: one proves geometrical formulas or relations for the positive values of a variable, for finally applying the result to a negative value. In this correspondence, the equalities are generally preserved, but not the inequalities.

We will be satisfied to introduce it in dimension 2 (the pseudo-Euclidean plane geometry), because it is easier to apply the visual intuition of geometry there, and it is enough to make most of the work to present the differences with Euclidean geometry. Then, going up to dimension 3 or 4 would be a difficult exercise for intuition but it would be secondarily useful, as from an abstract point of view this generalization is quasi automatic and includes only few new phenomena (which we will explain later; but in practice it is not often necessary to take into account the four dimensions at the same time).

The two other problems, the relation with the experience and mechanics, are based on the geometrical structures of space, therefore they depend on this geometry. However, they can be expressed in the same manner with any of all these geometries (Euclidean or pseudo-Euclidean, of any dimension), thanks to the fact that these geometries all have the same fundamental language in common (the same list of structures; only the properties of some of these structures have differences).

We will thus express them by making use of the geometrical structures in the language of Euclidean geometry. And indeed it is possible to interpret them within the framework of Euclidean geometry, in other words the one of our daily experience of fixed objects, to well understand their nature.

The fact that the expression of relativistic mechanics is the same as that of equilibrium can be explained simply: equilibrium is the situation of relativistic mechanics where the material systems do not undergo any evolution in a certain time direction; then, anything can still occur in the three remaining dimensions according to the normal laws of relativistic mechanics, up to the only difference that the time dimension (more precisely the time direction involved) does not intervene anymore. The law of equilibrium is thus the same one as relativistic mechanics with one less dimension, which is the result we were looking for.

3.2. Link with the experience

The question of the relation of this reality to our experience of space and time, is the only easy question, which does not require any significant mathematical work; it does not bring any information on the laws of physics themselves. The question is this one: how do we visit this world of dimension 4 where things are fixed, so that it appears to us in the form of a world of dimension 3 where things evolve ?

In general, this question applies to the visit of a world of dimension $n \geq 2$ where things are fixed (and for which time thus does not exist), that lets it appear as of a world of dimension $n-1$ where things evolve.

In this story, the time during which things appear to evolve is not an objective time, but it has only a subjective meaning, being lived separately for each observer, and not having thus a priori any other necessary relations with the others'time except those occuring through the material things which are fixed.

In fact, we already skimmed the response while introducing Euclidean geometry: physically, each observer (or his body or his spaceship or even the Earth, as its size is indeed negligible since it is less than one tenth of light second), is similar to a wire laid out in space on a curve called its *worldline*. The idea is that this wire as a space of dimension 1 plays the role of the personal time line of the observer. Thus, during his own time, he visits the world by going along his worldline at a constant speed, as a train would follow a railway at constant speed or as a light impulse follows an optical fibre. This speed of travel is assumed to be a universal constant v. We can numerically give it any value we want since, as time is here a subjective and not physical quantity (the physical objects cannot measure it since they are fixed), one can physically define its unit only in terms the length of curve followed at this speed.

An observer can, from his worldline, observe the worldline of another observer, not the other observer himself (because only the worldline is material). The feeling of acceleration is defined by the curve of this line in the same way as the centrifugal force in a car going at constant speed is defined by the curve of the road or in an equivalent way by the position of the wheel.

A clock is physically similar to an observer: it is also a wire laid out in space, but regularly graduated. Thus if the worldline of an observer is superimposed on that of a clock (he keeps the clock with him), he sees during his personal time the graduations of the clock come at a regular rythm, as a good clock should behave.

The paradoxes of relativity come from the geometrical effects which arise in their complexity but cut off from their geometrical explanatory intuition. One can check that the Galilean theory of space time is obtained as the limit of this model when the speed of travel v along worldlines approaches infinity; and compared to the Galilean theory, this model makes corrections whose first term has the range of order of v^{-2}. But, the work of changing the geometry that we will further make can be summarized (forgetting experiments involving light itself, which cannot be modelized this way) by the fact of giving this v^{-2} a negative value, according to the formula

$$v^2 = -c^2.$$

Thus the relativistic effects (expressed using c) are finally opposite to those which would be normally obtained from the above model in the case of the Euclidean geometry.

Would not the following geometrical effects, exactly opposite of the corresponding relativistic effects, be somehow "paradoxical" from a purely logical point of view in the absence of an explanatory visual support:

– When we see far away a truck progressing from left to right, why does it suddenly narrow when its way turns to us or away (relativistic dilation of time)?

– Why do its two nose gear wheels which were superimposed, separate then in our point of view (relativity of simultaneity)?

– Why does it take different times to go from a point to another following different paths, whereas we always follow them at the same speed (twins paradox)?

– Lastly, once defined the size of a sausage by the width of its discs, how can it be that it becomes larger when its axis is tilted with respect to the plane of cut than when it is orthogonal (relativistic contraction of lengths)?

Now we essentially completed this part. Let us add some more remarks.

An *inertial obsever* is an observer whose worldline is a straight line (not any straight line but one whose direction is temporal, i.e. that can be followed in time, which in pseudo-Euclidean geometry is different from a "space" direction), and whose point of view we will use to describe things; what is important here is the direction of this worldline, thus some fixed vector.

Vocabulary. *By convenience for the continuation of this presentation, we will use the qualifier of "theoretical" to indicate the use of the geometrical mode of representation for space-time, where things appear fixed (which can first be understood as if the geometry were Euclidean, then taking account of its pseudo-Euclidean nature). To that will be opposed the qualifier of "experimental" to indicate the description given by ordinary experimental appearances of fixity or nonfixity in 3-dimensional space that objects of relativistic mechanics can take with respect to a given inertial observer (where fixity qualifies the systems invariants by time translation along the direction of this observer's worldline).*

This somewhat twisted use of the opposite terms of "theory" and "experiment" should not lend to confusion since their normal use to which it competes will not appear in this presentation (where all is theoretical in the usual sense).

For example, the theoretical concept of point is experimentally called by the term of *event*, i.e. a point which for an observer appears only at one precise moment.

Remark 1. To the above description, two informations should be added that require the use of concepts specific to the pseudo-Euclidean geometry.

The first one is the condition so that a theoretical curve can be a worldline (of an observer or a particle). Indeed, in Euclidean geometry every curve could be the image of a wire (which makes it possible to define the affine parameter), which is not true any more in pseudo-Euclidean geometry. In particular, closed curve is not a possible worldline, so that one cannot go back in time. Precisely, the succession of points in a worldline must respect the *time order* (or order of causality): it is a partial order relation which is preserved by the group of the theoretical displacements.

It is obvious that such a relation of order cannot exist in Euclidean plane geometry, since central symmetry which is there a rotation of pi radians, would reverse it. But central symmetry in the pseudo-Euclidean plane is not a rotation.

The second thing is to define what the observer really observes: the image observed from an event of observation, is given by light, which arrives "at the speed of light", therefore coming from a set of events which forms a cone (with vertex the event of observation), called the cone of past light of this event. We will describe its geometrical properties later.

Remark 2. A surperficial defect of this vision is that it seems to carry a fatalistic conception of the world (philosophy according to which all future things would be fixed in advance before they occur and are just discovered), as well as a distinction between matter and spirit. But this is just a mere impression due to the use of a particular form of intuition chosen for its practical advantages for the understanding of the physical space, and which does not express any real argument in favour of one or the other of these philosophies.

3.3. Deformed study of the Euclidean plane geometry

We will present here the Euclidean geometry expressed through deformed modes of representation. Then the means here necessary to manage such an approach will allow us to "magically" come to the pseudo-Euclidean geometry.

This deformation applies, in an equivalent way, to the two usual modes of representation of geometry: figures (in the usual plane), and coordinate systems. Drawings will not accurately represent the properties of the geometrical figures to study, because compared to reality, they will appear crushed in a certain direction, as when they are seen skew, from a distant point of view. One can then trust appearances of the drawing only as concerns the concepts of affine geometry of the plane. The other concepts must be rebuilt, in a way that does not respect what is seen on the drawing.

Circles

A circle has on the drawing the aspect of an ellipse, with a large axis and a small axis, which are orthogonal lines (as well actually as on the drawing; their directions are the only pair of directions having this property). The direction of the small axis and the ratio measured on the drawing

$$k = \frac{\text{length of the small axis}}{\text{length of the large axis}}$$

depend on the point of view used to make the drawing, but do not depend on the circle.

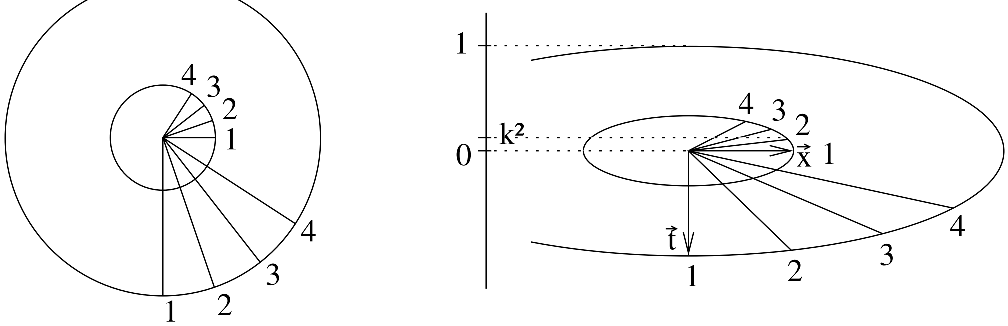
Note: starting from any ellipse in a drawing, one can suppose that it represents a circle. Then, having chosen an ellipse to play the role of a circle, the other circles are defined to be the ellipses obtained from this one by homotheties and translations.

Orthogonal frames

We will use these axes to form a frame of reference, orthonormal on the drawing (indeed, using the Euclidean geometry of the drawing one can copy the apparent unit of length from an axis to the other), but only orthogonal in reality. The coordinates will be noted (t, x), where the *time axis* (of equation $x = 0$) is the small axis of the ellipse, and the *space axis* (of equation $t = 0$) is the large axis (t is the time coordinate and x is the space coordinate). Let us note $(O, \vec{t}, \vec{x})$ this frame. The vectors $\vec{t}$ and $\vec{x}$ seem both to be of norm 1 on the drawing, but in fact $||\vec{x}|| = k||\vec{t}|| = 1$.

This purely artificial use of the space and time vocabulary to indicate coordinates is only justified here by the fact that a point following the "time axis" at the speed v will have apparent speed equal to 1 on the drawing, so that the "time coordinate" on this axis is identified to the time of this travel. So we have $v = k^{-1}$.

Then, starting from such a main reference frame, the other frames we will use will be those which are really orthogonal and have the same values of the norms of the basis vectors (and thus the same value of k) without taking account the appearances on the drawing, in other words the frames obtained by true rotations starting from a fixed main reference frame.



Real figure studied — Drawing used (expanded here)

Distances

We will not consider the apparent distances on the drawing, as they do not have a natural relationship with the studied reality. We will consider measurements of the real distances, but according to two possible units of length, which are the real lengths of the two main basic vectors. Precisely, the unit of space is worth k times the unit of time. Thus, the measurement in unit of time of the norm of a vector u of coordinates (t, x) is given by

$$||u||_t = \sqrt{t^2 + k^2x^2} = \sqrt{t^2 + \frac{x^2}{v^2}}$$

and is the apparent length on the drawing, of the vector obtained by a true rotation of u to bring it on the time axis, and in a similar way its measurement in unit of space is given by

$$||u||_x = k^{-1}||u||_t = \sqrt{v^2t^2 + x^2}.$$

To each of these units corresponds in the same way a measurement of the scalar product: $u \cdot u' = tt' + k^2xx'$ or $v^2tt' + xx'$.

Thus, k will be treated as a fixed number. However, without treating it in a different way, and since its actual value does not explicitly intervene in formal calculations, one can more judiciously interpret it as being not a real number but a quantity in the study in coordinates: the measurements of distances in units of time and space are then interpreted not as real numbers but as independent quantities (of times and lengths), and the quantity v or its reverse k is then the universal constant which establishes the natural physical link between them.

Angles

Angles are defined by the ratio of length of an arc of circle over its radius. However, here, the radius will be measured in unit of time, and the length of the arc of circle in unit of space. The unit of angle used here will be thus equal to k radians. As the functions cos and sin use as they should the measurement of the angles in radians, the rotation of an angle α expressed in this new unit will thus send the time vector $\vec{t} = (1, 0)$ onto the vector of components $(\cos(k\alpha), k^{-1}\sin(k\alpha))$, and the space vector $\vec{x} = (0, 1)$ onto $(-k\sin(k\alpha), \cos(k\alpha))$.

Note that as long as we are never interested in the intersections of the same circle with the two axis of a frame, nor with rotations of a right angle, the coefficient k and the trigonometrical functions do not appear in an independent way, but one can always gather their appearances in blocks of $\cos(k\alpha)$, $k^{-1}\sin(k\alpha)$ and k^2. (Or more generally, the functions of k which intervene are analytical functions of k^2). *We will exclude any geometrical concept in which they do not appear grouped this way.*

3.4. Passage to the pseudo-Euclidean geometry

From the deformed study of Euclidean geometry, we can get to the study of pseudo-Euclidean geometry (of which any representation is necessarily deformed), just by letting k^2 negative.

Precisely, the pseudo-Euclidean geometry, where the natural correspondence between measurements of space and time is expressed by the speed of light c, is obtained starting from the above model of deformed study by the formula

$$v^2 = -c^2 \quad (= k^{-2})$$

which implies that

$$\cos(k\alpha) = \cos(\frac{\alpha}{v}) = \mathrm{ch}(\frac{\alpha}{c})$$
$$k^{-1}\sin(k\alpha) = v\sin(\frac{\alpha}{v}) = c\ \mathrm{sh}(\frac{\alpha}{c})$$

where the functions *hyperbolic cosine* ch and *hyperbolic sine* sh are defined by

$$\mathrm{ch}\, u = \frac{e^u + e^{-u}}{2}, \qquad \mathrm{sh}\, u = \frac{e^u - e^{-u}}{2}.$$

(These expressions can also be obtained geometrically from calculations in the light frames presented later).

Inequalities which were true in Euclidean geometry it are generally no more true here. In particular, the scalar product continues to exist but the scalar square is not always any more positive.

Contrary to the preceding case, to measure the norm of a given vector $\vec{u}(t, x)$ one cannot choose any more arbitrarily between the unit of time and that of space, but this choice is determined by the *type* of vector, i.e. by the sign of its scalar square, let us say here (the one in unit of time squared) $t^2 - c^{-2}x^2$: if it is positive, the vector $\vec{u}$ is time-like and $||\vec{u}||$ can only be measured in unit of time; if it is negative, $\vec{u}$ is space-like and $||\vec{u}||$ can only be measured in unit of space; if it is null although $\vec{u} \neq 0$, $\vec{u}$ is an *isotropic* vector or light vector (its direction is a possible direction of a particle of light in space-time) and $||\vec{u}|| = 0$.

While fixing the origin of the plane (or better by looking at the set of vectors, set which forms a plane), the vectors of light form two secant lines which divide the plane into four parts, one of which consists of the isotropic or time-like vectors such that $t \geq 0$, and the null vector: they are the *future vectors*.

More generally, this concept can be defined in larger dimension: in a frame of reference, let $\vec{u} = (x, y, z, t)$. Its scalar square counted in unit of time squared is

$$\vec{u}^2 = t^2 - c^{-2}(x^2 + y^2 + z^2).$$

The vector $\vec{u}$ is called a *future vector* if in this sense $\vec{u}^2 \geq 0$ and $t \geq 0$.

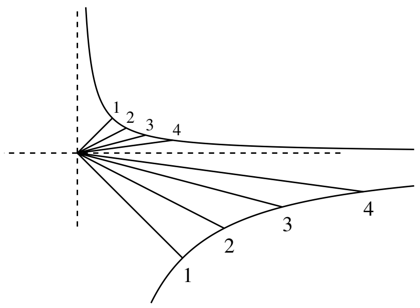
One can now define the relation of *time order* between the points of the pseudo-Euclidean plane: given two points M and N, we say that $M \leq N$ (M is former to N) if and only if $\overrightarrow{MN}$ is a future vector. One can check that this is indeed an order relation. If $M \leq N$, the time interval (distance in temporal unit) from M to N is $\sqrt{\overrightarrow{MN}^2}$.

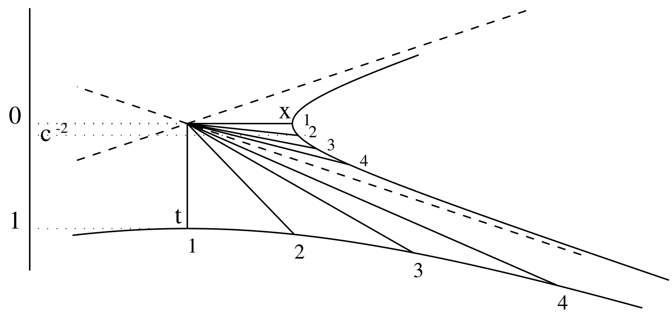
A curve is a possible worldline provided that the above defined time order relation between the points of the curve is a total order; this order coincides then with the order of the time lived by the visitor who follows this line. In fact, this condition is equivalent to require that the length of this curve be a well defined quantity. For two points $M \leq N$, among the possible worldlines from M to N, the straight line is the longest worldline and the only one whose length in time measure is equal to the interval of time from M to N.

The shortest worldlines have length zero and are made of segments with directions of light, but they are not possible worldlines for observers or any object of nonnull mass, which cannot reach the speed of the light: for the worldline of a photon the tangent vectors have null scalar square while for that of a particle of nonnull mass or of an observer, all the tangent vectors are strictly of time type (of nonzero scalar square).

The circles of this geometry are no more particular ellipses but particular hyperbolas, of two possible kinds (the kind of a circle being the type of its radius). Those of the same kind are still those obtained from one of them (that can be chosen arbitrarily) by homotheties and translations. All circles (of the two kinds) are the hyperbolas having the same pair of directions of the asymptotes, which are the light directions in this geometry. Each circle has two branches with infinite length each, since the length of an arc of circle is proportional to the angle which is also an angle of a rotation, and rotations can always be composed (adding the angles) without ever looping back.

Two lines are orthogonal if and only if one can map one to the other by a symmetry with respect to a light line parallel to the other light direction. The light (isotropic) vectors are those orthogonal with themselves.





The light reference frames

In the pseudo-Euclidean plane there is a possible new tool of study as compared to the Euclidean geometry: the use of a reference frame whose two axis follow the directions of light. By noting (a, b) the coordinates in such a reference frame, the rotation of an angle α around the origin is expressed by

$$\begin{cases} a' = & a \exp(\frac{\alpha}{c}) \\ b' = & b \exp(-\frac{\alpha}{c}) \end{cases}$$

One can notice the similarity with the use of complex numbers in Euclidean geometry, where the rotation of an angle α is expressed by the multiplication by $\exp(i\alpha)$. Indeed, in coherence with the definition of the new unit of angles which we introduced, let us consider now the problem in unit of time, and represent the Euclidean plane (x, y) by space-time coordinates (t, x') defined by $(t = x,\ x' = k^{-1}y)$. This deformation of the coordinates is reflected on the expression of complex numbers: we replace the use of i by that of some i' to still write the "complex number" in the form $z = x + iy = t + i'x'$, thus $i'^2 = -k^2 = c^{-2}$.

As the square of i' becomes positive for pseudo-Euclidean geometry, this formula thus admits two real solutions $i' = \pm c^{-1}$ which correspond to the two coordinates in a light reference frame. It is thus the fact of getting the possibility to look at the solutions of this equation (instead of preserving it just as it is during calculations) which is new compared to the Euclidean geometry, and allows, by imitating the use of complex numbers as a tool for Euclidean geometry, to make certain calculations somehow simpler or more explicit.

It is easy to see that the coefficient $\exp(\frac{\alpha}{c})$ is the measure of the Doppler effect (the number by which the frequency of an electromagnetic wave is multiplied during this rotation of the pseudo-euclidean plane).

The time order $M \leq N$ is expressed in a light reference frame by the same order on each cooordinate: ($a_M \leq a_N$ and $b_M \leq b_N$).

Example of concrete calculation

Problem

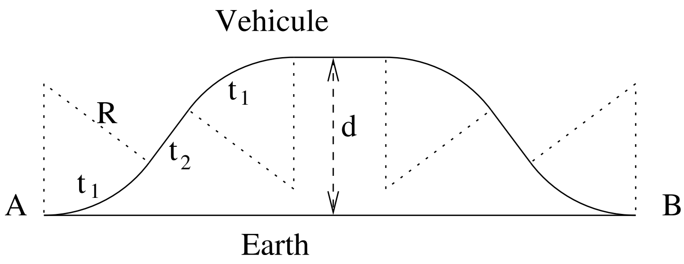
Imagine a traveller who leaves the Earth by doing from his own point of view an acceleration g for a duration t_1, then continues on his impetus without acceleration for a duration t_2, and finally boosts back by the same acceleration as in the beginning in opposite direction to stop with respect to the Earth. At which distance d from the Earth is he then? And if then he returns back to Earth in the same way, of how much will be the difference δt between the duration of his travel for him and for somebody remained motionless on the Earth?

(Let us recall that, for not being wrong to deal with this problem by special relativity as we do here without requiring general relativity, it is necessary to neglect in a certain way the terrestrial gravity, more precisely the speed of exhaust of the terrestrial field of gravitation).

Solution

Let us first translate the statement of this problem into the language of Euclidean geometry.

Starting from a straight road called "Earth", a vehicle (going at a high constant speed $v = k^{-1}$) starts a turn, the wheel being put in a certain position which makes it feel a lateral push g for a duration t_1. Then the wheel is rectified, and thus remains for a duration t_2. Finally, by directing it in the other direction in the same way one gets back after still a duration t_1 to rectify the vehicle to another straight road parallel to the Earth. At which distance from the Earth is one then? If one returns in the same way, which is the difference in duration taken to reach there the point of arrival B from the starting point A, compared to a vehicle which would have connected them while remaining on the straight road "Earth"?



During the turn, the vehicle follows an arc of circle of length t_1 in unit of time or vt_1 unit of space, with angle $\alpha = t_1 g$ (which gives in radians $\alpha k = t_1 g v^{-1}$) and of radius (in unit of space) $R = v^2 g^{-1}$.

We obtain by an elementary geometrical observation

$$d = 2R(1 - \cos(k\alpha)) + vt_2 \sin(k\alpha) = 2v^2 g^{-1}(1 - \cos(kt_1 g)) + vt_2 \sin(\frac{t_1 g}{v})$$

thus by using the "magic formula" $v^2 = k^{-2} = -c^2$ we obtain the result of the problem of relativity

$$d = 2c^2 g^{-1}(\mathrm{ch}(\frac{t_1 g}{c}) - 1) + ct_2 \,\mathrm{sh}(\frac{t_1 g}{c}).$$

In the same way we calculate the difference in length in unit of time between the curve and the straight line:

$$\Delta t = 4(t_1 - kR\sin(k\alpha)) + 2t_2(1 - \cos(k\alpha)) = 4(t_1 - cg^{-1}\,\mathrm{sh}(\frac{t_1 g}{c})) + 2t_2(1 - \mathrm{ch}(\frac{t_1 g}{c}))$$

which is finally a negative quantity, the straight line being longer than the curved line as announced.